

# *Fractal-Based Point Processes*

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## *Fractal Shot Noise*



While engaged in studies under Max Planck in Berlin, the German physicist **Walter Schottky (1886–1976)** characterized the stochastic properties of the random current arising from irregular electron arrivals at an anode; he bestowed on this process the name “shot noise.”



**Steven O. Rice (1907–1986)**, an American electrical engineer, studied the mathematical properties of shot noise in fine detail, demonstrating that its amplitude probability density often approaches a Gaussian form as the driving rate of the process increases without limit.

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Fractal shot noise, like a fractal Gaussian process, is a continuous-time process. Since it is everywhere nonnegative, it can serve as the rate of a doubly stochastic Poisson process, or an integrate-and-reset process, thereby generating associated point processes. Because many characteristics of these point processes derive from the properties of the underlying continuous rate process, we devote this chapter to the properties of fractal shot noise. Extensive discussion of the ensuing fractal-shot-noise-driven point processes is provided in Chapter 10. The material presented in this chapter and the next derives principally from Lowen & Teich (1989a,b, 1990, 1991).

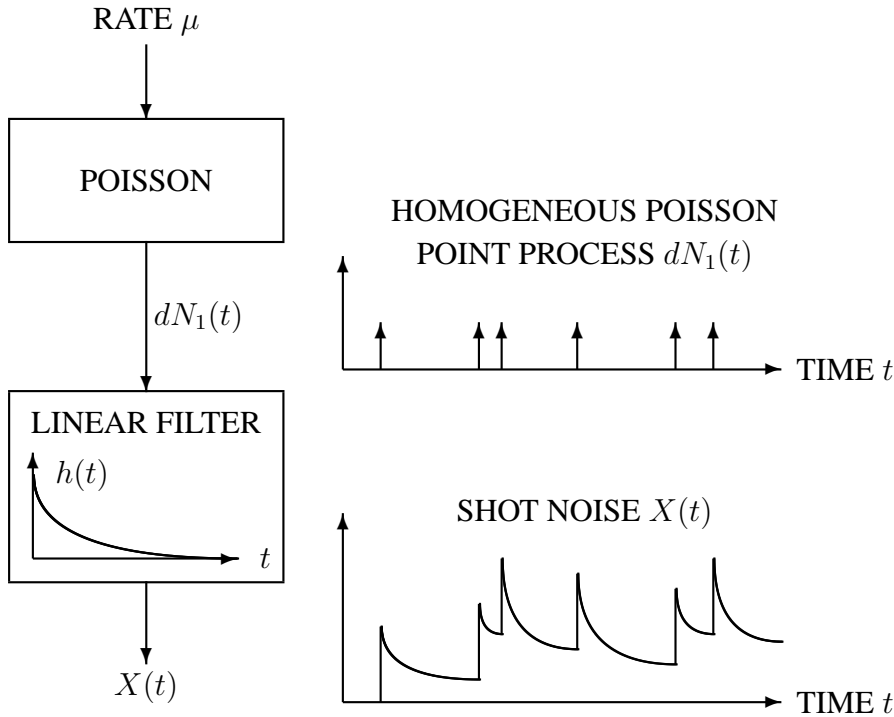
The mean and variance of the classic shot-noise process were established by Campbell (1909b,a) at the beginning of the last century. Not long thereafter, Walter Schottky (1918) defined and extensively studied this process, and bestowed on it the name “shot effect.” Twenty five years later, Rice (1944, 1945) carried out a classic detailed study of shot noise in which he demonstrated an important general feature of the process: the probability density function of its amplitude usually approaches a Gaussian form when the impulse response function has a finite duration and the emissions are dense. The classic shot-noise process serves as a remarkably useful construct in many fields of endeavor and it has been extensively studied (see, for example, Rice, 1944, 1945; Gilbert & Pollak, 1960; Picinbono, 1960; Saleh & Teich, 1982; Davenport & Root, 1987; Papoulis, 1991; Lax, 1997).

As schematically illustrated in Fig. 9.1, **shot noise** results from driving a memoryless, linear filter by a train of impulses derived from a homogeneous Poisson point process. The constant rate of event production  $\mu$  characterizes the homogeneous Poisson process, and the **impulse response function**  $h(t)$  characterizes the linear filter.

## 9.1 SHOT NOISE

As indicated in the schematic provided in Fig. 9.1, we define the shot-noise amplitude  $X(t)$  in terms of an infinite sum of impulse response functions. The impulse response functions themselves are assumed to be deterministic. They can have stochastic components, however, so that we write

$$X(t) \equiv \sum_{k=-\infty}^{\infty} h(K_k, t - t_k). \quad (9.1)$$



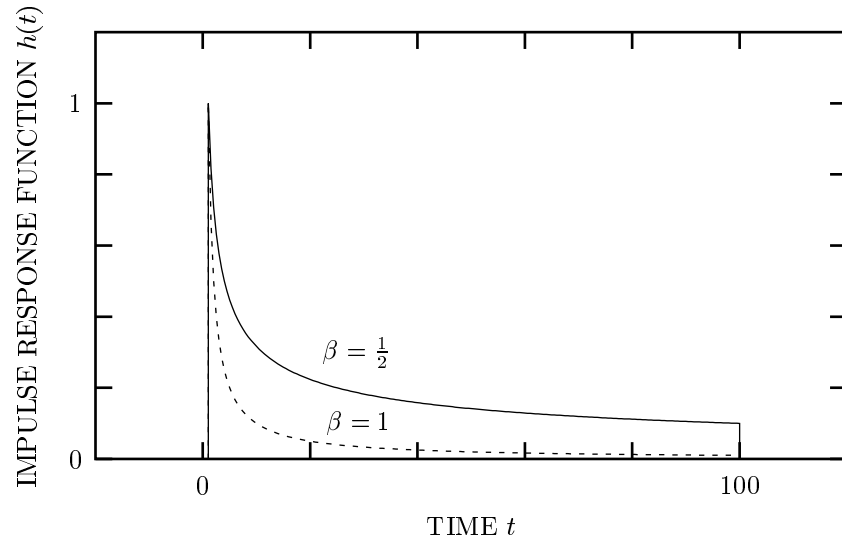
**Fig. 9.1** A linearly filtered Poisson point process gives rise to shot noise. The quantity  $\mu$  represents the rate of the Poisson process,  $h(t)$  is the impulse response function of the linear filter, and  $X(t)$  is the shot-noise amplitude. Fractal shot noise results when  $h(t)$  takes the form of a decaying power-law function.

The random event times  $t_k$  belong to a homogeneous Poisson point process of rate  $\mu$ , and  $\{K_k\}$  is a random sequence that serves as an index for the impulse response functions  $h(K, t)$ . We take the elements of the random sequence  $\{K_k\}$  as identically distributed, and independent of each other and of the Poisson process. Shot noise endowed with such an additional degree of randomness in its impulse response function (see, for example, Gilbert & Pollak, 1960; Picinbono, 1960) is known as **generalized shot noise**.

**Fractal shot noise** forms an important special case (Lowen & Teich, 1989b,a, 1990) in which the impulse response function assumes a general decaying power-law form,

$$h(K, t) \equiv \begin{cases} Kt^{-\beta} & A < t < B \\ 0 & \text{otherwise,} \end{cases} \quad (9.2)$$

as portrayed in Fig. 9.2. We refer to this process as *fractal* shot noise because power-law functions often characterize one or another of its properties in addition to the



**Fig. 9.2** Power-law impulse response functions  $h(t)$  vs. time  $t$ , with lower cutoff time  $A = 1$ , upper cutoff time  $B = 100$ , deterministic amplitude  $K = 1$ , and two decay exponents:  $\beta = \frac{1}{2}$  (solid curve) and  $\beta = 1$  (dashed curve).

impulse response function; these properties include the amplitude probability density, autocorrelation, and spectrum.

The parameters  $A$ ,  $B$ , and  $\beta$  are deterministic, fixed, and nonnegative. In general, the range of the impulse response function may extend down to  $A = 0$  and up to  $B = \infty$ , and  $\beta$  may take any finite positive value. The formalism presented here assumes that all component impulse response functions have the same duration and power-law shape, but need not have the same amplitudes  $K$ . We consider a more general multifractal version of this impulse response function in Sec. 13.3.8.

Markedly different behavior obtains for different ranges of the parameters, as delineated in Table 9.1. For some parameters, the process exhibits a  $1/f$ -type spectrum. For a square-integrable impulse response function, the process converges to a Gaussian form by virtue of the central limit theorem. Conversely, when the impulse response function has infinite tail area, the resulting shot-noise process assumes a value of infinity with probability one; in fact, a degenerate process results even when the impulse response function is normalized to constant area (see Sec. A.6.1). On the other hand, impulse response functions with infinite area near the origin result in a process that is not degenerate but rather follows ( $B = \infty$ ) or approaches ( $B < \infty$ ) a stable distribution with infinite mean.

In the fractal shot-noise processes considered above, the impulse response functions themselves vary in a power-law fashion, as provided in Eq. (9.2). Another version of fractal shot noise may be constructed by endowing relatively simple impulse response functions, such as those that are rectangular, with variable duration

	$A > 0$ $B = \infty$	$A = 0$ $B = \infty$	$A = 0$ $B < \infty$	$A > 0$ $B < \infty$
$0 < \beta < \frac{1}{2}$ ( $2 > \alpha > 1$ )	$\Pr\{X = \infty\} = 1$ no $S(f)$		$X \rightarrow \text{Gaussian}$ $S \sim 1/f^\alpha$	
$\frac{1}{2} \leq \beta < 1$ ( $1 \geq \alpha > 0$ )			$E[X^2] = \infty$ $S \sim 1/f^\alpha$	
$\beta = 1$			$E[X] = \infty$ no $S(f)$	
$\beta > 1$ ( $0 < \zeta < 1$ )	$X \rightarrow \text{Gaussian}$ $S(f)$ not $1/f^\alpha$	$X = \text{stable}$ no $S(f)$	$X \rightarrow \text{stable}$ no $S(f)$	$S(f)$ not $1/f^\alpha$

**Table 9.1** Characteristics of the amplitude probability density functions and spectra for fractal shot noise with various values of the parameters  $A$ ,  $B$ , and  $\beta$ . A variety of different features emerge. For  $B = \infty$  and  $\beta \leq 1$ , the impulse response function has a tail of infinite area; this yields a degenerate process for which finite amplitude values do not occur. Other regions in the parameter space yield well-defined shot-noise processes  $X(t)$ , although some have infinite variance or infinite mean. In particular, for  $A = 0$  and  $\beta > 1$ , the distribution of  $X$  either is ( $B = \infty$ ) or approaches ( $B < \infty$ ) a stable random variable with parameter  $\zeta$ , assuming that  $E[K^\zeta]$  exists. In other regions, both the mean and variance exist, and  $X$  converges to a Gaussian random variable as  $\mu \rightarrow \infty$ , assuming that  $E[K^2]$  exists. Finally, for  $B < \infty$  and  $\beta < 1$ , the shot-noise process exhibits a spectrum that decays as  $1/f^\alpha$ .

and amplitude, and ascribing decaying power-law distributions to these parameters (see Prob. 9.2). Although distinct in their construction, the two formalisms yield similar results (Ryu & Lowen, 2000; Masoliver, Montero & McKane, 2001). Indeed, with a proper choice of distributions their amplitude statistics can be made to coincide (Gilbert & Pollak, 1960; Picinbono, 1960; Lowen & Teich, 1990). This variant of fractal shot noise is useful in modeling computer network traffic (see Chapter 13).

### 9.2 AMPLITUDE STATISTICS

Standard shot-noise theory provides the characteristic function  $\phi_X(\omega)$  of the shot-noise process  $X$  (Doob, 1953; Saleh & Teich, 1982; Davenport & Root, 1987):

$$\phi_X(\omega) \equiv E[e^{-i\omega X}] = \int_K \exp\left(-\mu \int_{t=-\infty}^{\infty} \left\{1 - \exp[-i\omega h(K, t)]\right\} dt\right) p_K(y) dy, \tag{9.3}$$

where  $p_K(y)$  represents the probability density function of the impulse-response-function amplitudes  $K$ . For the specific case of a power-law decaying impulse response function, as provided in Eq. (9.2), with deterministic amplitudes  $K$ , we obtain

$$\begin{aligned}
 \ln[\phi_X(\omega)] &= -\mu \int_A^B [1 - \exp(-i\omega K t^{-\beta})] dt \\
 &= -\frac{\mu (i\omega K)^{1/\beta}}{\beta} \int_{i\omega K B^{-\beta}}^{i\omega K A^{-\beta}} \frac{1 - e^{-u}}{u^{1+1/\beta}} du \\
 &= \mu A [1 - \exp(-i\omega K A^{-\beta})] \\
 &\quad - \mu B [1 - \exp(-i\omega K B^{-\beta})] \\
 &\quad + \mu (i\omega K)^{1/\beta} \Gamma(1 - 1/\beta, i\omega K A^{-\beta}) \\
 &\quad - \mu (i\omega K)^{1/\beta} \Gamma(1 - 1/\beta, i\omega K B^{-\beta}).
 \end{aligned} \tag{9.4}$$

Here  $\Gamma(x, a)$  represents the incomplete Eulerian gamma function

$$\Gamma(x, a) \equiv \int_a^\infty t^{x-1} e^{-t} dt, \tag{9.6}$$

and integration by parts yields Eq. (9.5) from Eq. (9.4).

Derivatives of the logarithm of this function lead to the cumulants  $C_n$  of  $X$  (Rice, 1944, 1945), as defined in Eq. (3.8),

$$\begin{aligned}
 C_n &\equiv i^n \frac{d^n}{d\omega^n} \ln[\phi_X(\omega)]_{\omega=0} \\
 &= \mu \mathbb{E} \left[ \int_{-\infty}^{\infty} h^n(t) dt \right] \\
 &= \mu \mathbb{E}[K^n] \int_A^B t^{-n\beta} dt,
 \end{aligned} \tag{9.7}$$

which become (Lowen & Teich, 1990)

$$C_n = \mu \mathbb{E}[K^n] \times \begin{cases} \frac{A^{1-n\beta} - B^{1-n\beta}}{n\beta - 1} & \beta \neq 1/n \\ \ln(B/A) & \beta = 1/n. \end{cases} \tag{9.8}$$

The  $n$ th cumulant assumes an infinite value if  $\mathbb{E}[K^n]$  does, or if  $A = 0$  and  $\beta \geq 1/n$ , or if  $B = \infty$  and  $\beta \leq 1/n$ . The moments and cumulants determine each other, as shown in Eq. (3.9).

Two general approaches are available for obtaining the probability density  $p_X(x)$  for the shot-noise-process amplitude  $X$ . The first method involves carrying out the inverse Fourier transform of Eq. (9.3) [in a form much like Eq. (9.15)], which rarely proves feasible. The second method involves constructing an integral equation (Gilbert & Pollak, 1960). Note that if  $B < \infty$ , then  $\Pr\{X = 0\} = \exp[-\mu(B - A)] > 0$ , and thus the density has a delta function at  $x = 0$ .

For deterministic  $K$ , the amplitude probability density function then follows the integral equation (Lowen & Teich, 1989a, 1990):

$$p_X(x) = \begin{cases} 0 & x < 0 \\ \exp[-\mu(B - A)] \delta(x) & x = 0 \\ 0 & 0 < x \leq KB^{-\beta} \\ \frac{\mu K^{1/\beta}}{\beta x} \int_{KB^{-\beta}}^x p_X(x - u) u^{-1/\beta} du & KB^{-\beta} < x < KA^{-\beta} \\ \frac{\mu K^{1/\beta}}{\beta x} \int_{KB^{-\beta}}^{KA^{-\beta}} p_X(x - u) u^{-1/\beta} du & x \geq KA^{-\beta}. \end{cases} \quad (9.9)$$

If  $B = \infty$ , then Eq. (9.9) simplifies to

$$p_X(x) = \frac{\mu K^{1/\beta}}{\beta x} \int_0^{\min(x, KA^{-\beta})} p_X(x - u) u^{-1/\beta} du, \quad (9.10)$$

where  $\min(x, y)$  returns the smaller of  $x$  and  $y$ . The integral equation Eq. (9.10) admits a family of solutions, all proportional to each other; imposing the requirement that  $\int_0^\infty p_X(x) dx = 1$  provides the correct one.

For finite  $C_1$  and  $C_2$ , the amplitude probability density function  $p_X(x)$  satisfies the conditions of the central limit theorem, and therefore approaches a Gaussian density as  $\mu \rightarrow \infty$ . This always obtains for  $A > 0$  and  $B < \infty$  [see Eq. (9.7)], as displayed in the right-most column of Table 9.1. The first and second cumulants provide the mean and variance, respectively, of the resulting amplitude density, so that the limiting form becomes

$$p_X(x) \rightarrow (2\pi C_2)^{-1/2} \exp\left[-\frac{(x - C_1)^2}{2C_2}\right]. \quad (9.11)$$

In fact, the vector  $\{X(t_1), X(t_2), \dots, X(t_k)\}$ , for any positive integer  $k$  and any set of times  $\{t_1, t_2, \dots, t_k\}$ , possesses a joint Gaussian distribution, so that  $X(t)$  becomes a Gaussian process as  $\mu \rightarrow \infty$ . For finite  $\mu$  and for values of  $x$  close to the mean of the process ( $C_1$ ), we can expand the amplitude probability density about the asymptotic Gaussian result (Rice, 1944, 1945), which yields (Lowen & Teich, 1990)

$$p_X(x) \approx (2\pi C_2)^{-1/2} \exp\left[-\frac{(x - C_1)^2}{2C_2}\right] \times \left[1 - \frac{C_3}{2C_2^2} (X - C_1) + \frac{C_3}{6C_2^3} (X - C_1)^3\right]. \quad (9.12)$$

For  $\beta > 1$ ,  $A \rightarrow 0$ , and  $B \rightarrow \infty$ , a much simpler form obtains directly from the characteristic function (Lowen & Teich, 1990) (see Sec. A.6.2). The resulting expression, written as

$$\phi_X(\omega) = \exp\{-\mu E[K^\zeta] \Gamma(1 - \zeta) (i\omega)^\zeta\}, \quad (9.13)$$



with  $\zeta \equiv 1/\beta$ , follows the general form

$$\phi(\omega) = \exp[-(ic\omega)^\zeta], \tag{9.14}$$

for a constant  $c$ . The shot noise  $X$  then has a one-sided stable distribution (Lévy, 1937, 1940; Pollard, 1946; Feller, 1971) for all  $\mu$ , with an associated parameter  $\zeta$  that lies between zero and unity. Stochastic values of  $K$  that assume positive and negative values make stable distributions with other than one-sided forms possible (Petropulu, Pesquet, Yang & Yin, 2000). The Gaussian and other stable distributions share the property that two random variables taken from the same distribution, and added together, result in a new random variable whose distribution differs from the original one only by a scaling constant; the Gaussian differs from the stable distributions encountered here in that the latter have infinite means.

Further, if  $A = 0$  and  $\beta > 1$ , but  $B < \infty$ , the amplitude  $X$  converges to a stable random variable as  $\mu \rightarrow \infty$  (see Sec. A.6.2). The vector  $\{X(t_1), X(t_2), \dots, X(t_k)\}$  for any integer  $k > 1$  and any set of times  $\{t_1, t_2, \dots, t_k\}$  does not have a joint stable distribution, so  $X(t)$  is not a stable process. However,  $X(t_1)$  and  $X(t_2)$  do become jointly stable as the separation  $t_2 - t_1$  approaches infinity (Petropulu et al., 2000).

Two methods exist for obtaining the associated amplitude probability density function  $p_X(x)$ . The first involves the use of a Fourier integral (Rice, 1944, 1945; Feller, 1971):

$$p_X(x) = \frac{1}{\pi} \operatorname{Re} \int_0^\infty \exp\{i\omega x - \mu E[K^\zeta] \Gamma(1 - \zeta) (i\omega)^\zeta\} d\omega, \tag{9.15}$$

whereas the second makes use of an infinite sum (Humbert, 1945; Pollard, 1946; Feller, 1971):

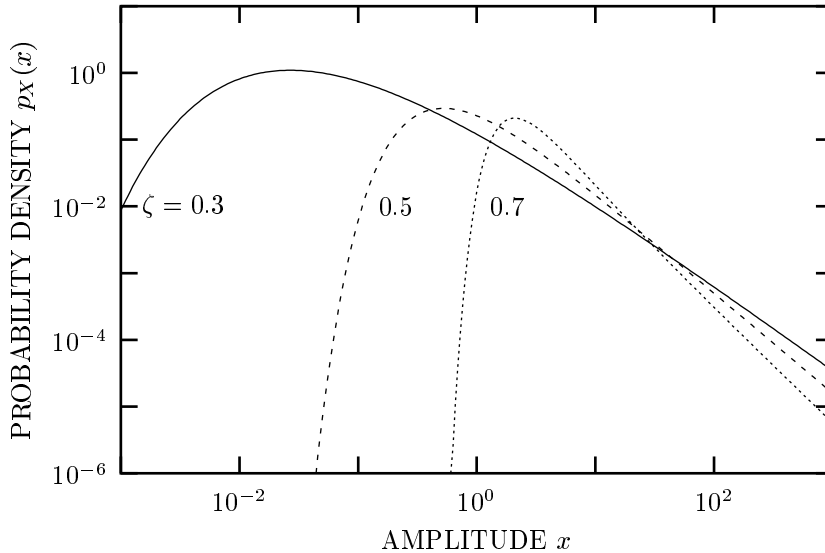
$$p_X(x) = \frac{1}{\pi x} \sum_{n=1}^\infty \frac{(-1)^{n+1} \Gamma(1 + n\zeta) \sin(\pi n\zeta)}{n!} \left[ \frac{\mu \Gamma(1 - \zeta) E[K^\zeta]}{x^\zeta} \right]^n. \tag{9.16}$$

Grouping adjacent terms (of opposite sign) and simplifying improves the convergence properties of the sum (Weiss, Dishon, Long, Bendler, Jones, Inglefield & Bandis, 1994). For large values of  $X$  the sum converges quickly, whereas for small values of  $X$  the integral proves more useful.

For the particular case  $\zeta = \frac{1}{2}$ , the amplitude probability density function is described by the well-known closed-form expression (Lévy, 1940; Feller, 1971)

$$p_X(x) = \frac{\mu E[K^{1/2}]}{2} x^{-3/2} \exp\left(-\frac{\mu^2 \pi E^2[K^{1/2}]}{4x}\right), \tag{9.17}$$

which is identical to Eq. (3.13) with  $t_0 = (\mu^2 \pi/4) E^2[K^{1/2}]$ . Lévy distributions arise in many contexts, such as the gravitational field produced by a random distribution of masses in one dimension and the electric field at the growing edge of a quantum wire, among others (see, for example, Good, 1961; Parzen, 1962; Lowen & Teich, 1989a). The force acting on a star as a result of the gravitational attraction of neighboring stars obeys a three-dimensional, spherically symmetric  $\frac{3}{2}$ -stable distribution,



**Fig. 9.3** One-sided stable amplitude probability density  $p_X(x)$  vs.  $x$  provided in Eqs. (9.15)–(9.17), for three values of the associated parameter:  $\zeta = 0.3, 0.5,$  and  $0.7$  ( $A = 0, B = \infty, K = 1, \mu = 1$ ). Long power-law tails are present for all values of  $\zeta$ .

which is traditionally called the **Holtmark distribution** (Holtmark, 1919, 1924; Chandrasekhar, 1943).

Figure 9.3 displays stable amplitude probability density functions for three values of the parameter  $\zeta$ . All have power-law tails, as provided by Eq. (9.16). For large  $x$  the first term dominates; in the limit  $x \rightarrow \infty$ , after simplification we obtain (Lowen & Teich, 1990)

$$p_X(x) \approx \mu \zeta E[K^\zeta] x^{-(1+\zeta)}. \quad (9.18)$$

For other infinite-area impulse response functions the resulting shot noise has trivial amplitude properties. For  $0 < \beta \leq 1$  and  $B = \infty$ , the shot-noise process  $X$  assumes an infinite value with probability one (Lowen & Teich, 1990) (see Sec. A.6.1). This degenerate case arises because the infinite area of each impulse response function lies in its tail, which persists throughout time. These tails accumulate to produce an unbounded sum. Even normalizing the impulse response functions to constant area as  $B$  increases toward infinity yields a degenerate process, in this case one with zero variance (Lowen & Teich, 1991) (see Sec. A.6.1). In contrast, for  $\beta > 1$  the infinite area occurs only during an infinitesimal interval immediately following the onset of the impulse response function; this leads to a well-defined shot noise process  $X(t)$  in spite of the fact that the mean is infinite.

### 9.3 AUTOCORRELATION

The autocorrelation of the process  $X(t)$  assumes the simple form (Rice, 1944, 1945)

$$R_X(t) \equiv \mathbb{E}[X(s)X(s+t)] = \mathbb{E}^2[X] + \mu R_h(t), \quad (9.19)$$

where the autocorrelation of  $h(K, t)$  itself obeys the equation

$$\begin{aligned} R_h(t) &\equiv \mathbb{E} \left[ \int_{-\infty}^{\infty} h(K, s) h(K, s+|t|) ds \right] \\ &= \mathbb{E}[K^2] \int_A^{B-|t|} (s^2 + |t|s)^{-\beta} ds. \end{aligned} \quad (9.20)$$

When  $|t| \geq B - A$ ,  $R_h(t) = 0$  so that  $R_X(t) = \mathbb{E}[X]^2$ .

The integral in Eq. (9.20) becomes infinite for parameters in the following ranges:

$$\begin{aligned} \beta &\leq \frac{1}{2} \quad \text{and} \quad B = \infty, \\ \beta &\geq 1 \quad \text{and} \quad A = 0, \\ \beta &\geq \frac{1}{2} \quad \text{and} \quad A = 0 \quad \text{and} \quad t = 0; \end{aligned} \quad (9.21)$$

in those cases  $R_X(t)$  does not exist (Lowen & Teich, 1990).

For parameter values outside the ranges set forth in Eq. (9.21), we can develop useful approximations to  $R_h(t)$ , and therefore to  $R_X(t)$ . For  $\beta < \frac{1}{2}$  and  $A \ll |t| \ll B$ , we have

$$\begin{aligned} R_h(t) &= \mathbb{E}[K^2] \int_A^{B-|t|} s^{-\beta} (s+|t|)^{-\beta} ds \\ &\approx \mathbb{E}[K^2] \int_0^B s^{-2\beta} ds \\ &= (1-2\beta)^{-1} \mathbb{E}[K^2] B^{-2\beta}, \end{aligned} \quad (9.22)$$

where the approximations derive from the specified range of  $t$ , and from the domination of the integrand by the tail. Thus, for these values of  $\beta$  and this range of  $t$ , the autocorrelation essentially remains fixed with respect to  $t$ .

For  $\frac{1}{2} < \beta < 1$ , and the same range of  $t$ , neither  $A$  nor  $B$  is important so that

$$\begin{aligned} R_h(t) &= \mathbb{E}[K^2] \int_A^{B-|t|} s^{-\beta} (s+|t|)^{-\beta} ds \\ &\approx \mathbb{E}[K^2] \int_0^{\infty} s^{-\beta} (s+|t|)^{-\beta} ds \\ &= \Gamma(1-\beta) \Gamma(2\beta-1) [\Gamma(\beta)]^{-1} \mathbb{E}[K^2] |t|^{1-2\beta}; \end{aligned} \quad (9.23)$$

the exponent  $1-2\beta$  lies between  $-1$  and  $0$ . For  $\beta > 1$  and  $A \ll |t| \ll B$  we obtain

$$R_h(t) = \mathbb{E}[K^2] \int_A^{B-|t|} s^{-\beta} (s+|t|)^{-\beta} ds$$

$$\begin{aligned}
&\approx \mathbb{E}[K^2] \int_A^\infty s^{-\beta} |t|^{-\beta} ds \\
&= (\beta - 1)^{-1} \mathbb{E}[K^2] A^{1-\beta} |t|^{-\beta}, \tag{9.24}
\end{aligned}$$

where the approximations derive from the specified range of  $t$ , and from the domination of the integrand by the area near the origin. Closed-form expressions exist for particular values of  $\beta$ , and for large values of  $|t|$  when  $\beta > 1$  (see Sec. A.6.3).

#### 9.4 SPECTRUM

Carson's theorem (Carson, 1931) provides the spectrum  $S_X(f)$  of the fractal shot-noise process  $X$  in terms of  $\mu$  and the Fourier transform  $\mathcal{F}$  of the impulse response function defined in Eq. (9.2) (Rice, 1944, 1945). We focus on the domain  $\beta < 1$  and  $B < \infty$  where  $1/f^\alpha$  spectral behavior prevails. Denoting the Fourier transform  $\mathcal{F}$  by  $H(f)$ , we obtain (Lowen & Teich, 1990)

$$H(f) \equiv \mathcal{F}\{h(t)\} = K \int_A^B t^{-\beta} e^{-i2\pi ft} dt \tag{9.25}$$

$$= K [\Gamma(1 - \beta, i2\pi fA) - \Gamma(1 - \beta, i2\pi fB)] (i2\pi f)^{\beta-1}, \tag{9.26}$$

where  $\Gamma(x, a)$  again represents the (incomplete) Eulerian gamma function defined in Eq. (9.6).

For  $B < \infty$  and  $\mathbb{E}[K^2] < \infty$  the autocovariance  $R_X(t) - \mathbb{E}^2[X]$  has a finite integral. Carson's theorem then applies, which yields (Lowen & Teich, 1990)

$$S_X(f) = \mathbb{E}^2[X] \delta(f) + \mu \mathbb{E}[|H(f)|^2] \tag{9.27}$$

$$\begin{aligned}
&= \mathbb{E}^2[X] \delta(f) \\
&\quad + \mu \mathbb{E}[K^2] \left| \Gamma(1 - \beta, i2\pi fA) - \Gamma(1 - \beta, i2\pi fB) \right|^2 \\
&\quad \times (2\pi f)^{-2(1-\beta)}. \tag{9.28}
\end{aligned}$$

This spectrum appears as the solid curve in Fig. 9.4.

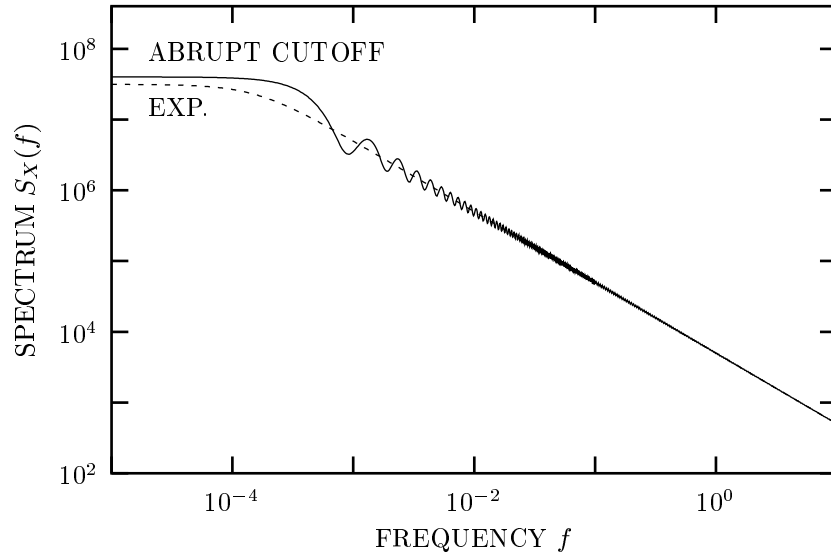
For  $0 < \beta < 1$ , fitting Eq. (9.28) with a spectrum of the form  $S(f) \sim f^{-\alpha}$  provides

$$\alpha = 2(1 - \beta). \tag{9.29}$$

In particular, for  $1/B \ll f \ll 1/A$ , and this range of  $\beta$ , the first and second incomplete gamma functions approach the complete gamma function and zero, respectively, whereupon

$$S_X(f) \approx \mu \mathbb{E}[K^2] \Gamma^2(\alpha/2) (2\pi f)^{-\alpha}. \tag{9.30}$$

The abrupt cutoff in the time domain leads to oscillations in the frequency domain (observe the solid curve in Fig. 9.4). The special case  $\alpha = 1$  was initially examined by Schönfeld (1955) and developed by van der Ziel (1979). Inspection of Eq. (9.25)



**Fig. 9.4** Spectrum  $S_X(f)$  vs. frequency  $f$  for fractal shot noise with different cutoffs: abrupt (solid curve) and exponential (dashed curve). The two processes do not otherwise differ, sharing the parameters  $\beta = \frac{1}{2}$ ,  $A = 0$ ,  $B = 1\,000$ ,  $K = 100$ , and  $\mu = 1$ . For sufficiently high frequencies, both curves exhibit  $1/f^\alpha$  behavior, with exponent  $\alpha = 1$ . The impulse response function with an abrupt cutoff gives rise to oscillations in the frequency domain whereas the exponential transition does not. Their low-frequency values approach different asymptotes since their mean values differ; Eq. (9.31) yields values of  $4 \times 10^7$  and  $10^7\pi \approx 3.14 \times 10^7$  for the abrupt-cutoff and exponential-cutoff impulse response functions, respectively.

reveals that the spectrum approaches a constant value in the limit  $f \rightarrow 0$ , for any impulse response function (Lowen & Teich, 1990):

$$\lim_{f \rightarrow 0} S_X(f) = \lim_{f \rightarrow 0} \mu E[|H(f)|^2] = \mu E[K^2] (E[X]/E[K])^2. \quad (9.31)$$

A power-law impulse response function with exponential transitions,

$$h_2(K, t) = K \exp(-A/t) \exp(-t/B) t^{-\beta}, \quad (9.32)$$

yields a smoother transition near the cutoff frequency  $f \approx (2\pi B)^{-1}$ , at the expense of more complex expressions for other quantities. In particular, for  $A \rightarrow 0$  the impulse response function is the same as that considered by Buckingham (1983, Chapter 6), which leads to (Lowen & Teich, 1989b, 1990)

$$\begin{aligned} H_2(f) &= K \int_0^\infty t^{-\beta} \exp(-t/B - i2\pi ft) dt \\ &= \Gamma(\alpha/2) B^{\alpha/2} K (1 + i2\pi fB)^{-\alpha/2} \\ S_{X_2}(f) &= E[X]^2 \delta(f) + \mu E[K^2] \Gamma^2(\alpha/2) B^\alpha [1 + (2\pi fB)^2]^{-\alpha/2}. \end{aligned} \quad (9.33)$$

In the high-frequency limit  $f \gg 1/B$ , Eq. (9.30) applies for this impulse response function as well. The spectrum for a shot-noise process with an exponential-transition impulse response function is displayed as the dashed curve in Fig. 9.4.

While Eq. (9.28) promises  $1/f$ -type behavior for  $\beta < 1$  and  $B = \infty$ , the process is actually degenerate for this set of parameters, as shown in Sec. A.6.1. Thus,  $1/f^\alpha$  behavior emerges only for  $\beta < 1$  and  $B < \infty$ . We note that long-duration memory exists in a generalized sense for other parameter ranges, notably for  $\beta > 1$  and  $A = 0$  (Petropulu et al., 2000).

We emphasize that the parameter  $\alpha$  is fundamentally different in character from the parameter  $\zeta$  for the stable probability distribution encountered in Eq. (9.13) and thereafter. The quantity  $\alpha$  describes the properties of a signal over time, with no reference to its amplitude. The quantity  $\zeta$ , on the other hand, characterizes only the amplitude distribution and is therefore unrelated to the time course of a signal. Considering a signal such as a fractal rate, the spectrum  $S(f)$  evaluated along the abscissa (horizontal axis) decays in a power-law fashion as  $f^{-\alpha}$ . Considering, now, the graph of a different signal with a stable amplitude distribution, the probability  $\Pr\{X > x\}$  of observing a large value  $x$  occurring on the ordinate (vertical axis) decays in a power-law fashion as  $x^{-\zeta}$ . Figuratively speaking, therefore,  $\alpha$  and  $\zeta$  describe orthogonal properties. Moreover, the stable distributions described here obey  $\zeta < 1$ , and therefore have infinite means and no spectra. No single process examined in the present work exhibits nontrivial values of both  $\zeta$  and  $\alpha$  simultaneously.

### 9.5 FILTERED GENERAL POINT PROCESSES

Although this chapter has been directed toward setting forth the properties of a linearly filtered homogeneous Poisson process, this particular focus does not represent a fundamental limitation of the approach. Indeed, the filtering of any orderly point process yields a well-behaved continuous-valued process, although the properties of such filtered general point processes are usually more difficult to derive when the underlying process is non-Poisson (Lukes, 1961). Filtered versions of many types of point processes have been examined (see, for example, Parzen, 1962; Weiss, 1973; Grandell, 1976; Snyder & Miller, 1991).

Our final considerations in this chapter are devoted to two measures for which tractable results for filtered general point processes are readily established: the mean and spectrum. We further restrict ourselves to linear filtering in which the random filter functions  $h(K, t)$  are independent of the point process  $dN(t)$ .

If  $\mu(t)$  represents the rate of the point process, the mean value of the resulting generalized process  $X(t)$  is simply

$$E[X] = E[\mu] E \left[ \int_0^\infty h(K, t) dt \right]. \tag{9.34}$$

The square of this quantity appears as a delta function in the spectrum  $S_X(\omega)$  of the generalized process  $X(t)$ . For other frequencies, linear systems theory (Papoulis,

1991) provides a spectrum given by

$$S_X(f) = E\left[|H(f)|^2\right] S_N(f). \quad (9.35)$$

These formulas are readily applied to the results provided in Chapter 4 for several classes of point processes. In the special case when  $dN(t)$  is a homogeneous Poisson process,  $S_N(f) = \mu$  and we recover Eq. (9.27).

### Problems

**9.1 Sums of fractal-shot-noise processes** Let  $X_m(t)$ , for  $1 \leq m \leq M$ , each denote an independent shot-noise process with rate  $\mu_m$  and impulse response function  $h_m(K, t)$ . Define  $X_R(t)$  as the sum of all the  $X_m(t)$ .

**9.1.1.** Suppose that the impulse response functions coincide for all  $m$ , so that  $h_m(K, t) = h_1(K, t)$  for all indices  $m$ . Show that  $X_R(t)$  also belongs to the shot-noise family of processes and find its rate  $\mu_R$ .

**9.1.2.** Now remove the identity among the impulse response functions. Assuming that  $k$  is fixed and deterministic within each component process  $X_m(t)$ , show that it is still possible to describe  $X_R(t)$  as a shot noise process.

**9.1.3.** If the  $X_m(t)$  are each fractal shot noise processes for all  $m$  with  $0 < \alpha < 2$ , under what conditions does  $X_R(t)$  belong to the fractal shot-noise family of processes?

**9.1.4.** Now suppose that  $h_m(K, t) = \exp(-c_m t)$ , where  $K$  is deterministic and set to unity for simplicity. Given this set of impulse response functions, is it possible to generate an approximation to fractal shot noise for  $0 < \alpha < 2$ ? If so, plot a representative spectrum.

**9.2 Rectangular impulse response function with power-law-distributed duration** As indicated at the end of Sec. 9.1, we need not limit the functional form of  $h(K, t)$  to a multiplicative decomposition such as  $Kh(1, t)$ . Consider the rectangular impulse response function

$$h(K, t) = \begin{cases} c & 0 < s < K \\ 0 & \text{otherwise.} \end{cases} \quad (9.36)$$

**9.2.1.** Setting  $c = 1$  for simplicity, find an expression for the autocorrelation of the resulting shot-noise process  $X(t)$ .

**9.2.2.** Now suppose that  $K$  has a probability density function that assumes the generalized Pareto form

$$p_K(t) = \begin{cases} (\beta - 1)A^{\beta-1} t^{-\beta} & t > A \\ 0 & t \leq A, \end{cases} \quad (9.37)$$

with  $2 < \beta < 3$ . Find an expression for the autocorrelation  $R_X(t)$  for  $t > A$ , and in particular, identify the associated range of  $\alpha$ .

**9.3 Decaying-power-law mass distributions** In many systems of aggregated particles the mass distribution  $P_{\mathcal{M}}(m)$  obeys a power-law form over some range of

masses  $m$ , such that (Witten & Sander, 1981; Grassberger, 1985; Takayasu, Nishikawa & Tasaki, 1988)

$$\Pr\{\mathcal{M} \geq m\} = cm^{-D}, \quad (9.38)$$

where  $c$  is a normalizing constant. The power-law exponent  $D$  typically falls in the range  $0 < D < 1$ . In some systems the number of particles in any given region is Poisson distributed and their masses are independent of each other and of the number of particles. Suppose we express the mass  $\mathcal{M}$  as the amplitude of a power-law impulse response function and then invert Eq. (9.38). Show how this procedure yields a fractal shot-noise process that provides appropriate values for the total mass in a specified region, and find the corresponding range of  $\beta$ .